Become an Escher Sleuth

Linda L. Cooper, Sandy M. Spitzer, and Ming C. Tomayko

Tessellations, the repetition of a shape that covers a plane without gaps or overlaps, are found throughout the world, from hexagons in honeycombs to rectangular bricks in masonry. For centuries, artists have tessellated tiles to decorate walls and walkways. However, there are arguably no tessellations more intriguing than the intricate designs of the Dutch artist Maurits Cornelis (M. C.) Escher. At first glance, Escher's works appear almost magical, and many ask, "How did he make these images?" The primary goal of this series of classroom activities is to enable students to decipher Escher's tessellations and identify the mathematics that he used to produce his artistic creations.

In 1922, and again in 1936, Escher traveled to the Alhambra in Granada, Spain, where he became captivated with Moorish tilings (see **figs. 1a–b**) that he found throughout the palace.

Fig. 1 Escher's inspiration came from Spain's Alhambra.

(a) (b)

Escher's study of these tilings revealed underlying mathematical properties of tessellations and opened a world of possibilities for his own artistic work. Many of the tessellations that Escher encountered in the Alhambra included more than one shape and sometimes included modifications of polygons (see **fig. 1**). However, all the tessellations that Escher found at the Alhambra consisted exclusively of geometric shapes without animal form, as was the custom of Islamic artists. Escher extended the mathematical ideas found in the artwork of the Moorish masters to create his own style of tessellations involving recognizable animal forms. One of his greatest "games" was to manipulate a polygon while maintaining its ability to tessellate, so that the resulting shape would be recognized as a lizard, bird, fish, horse, or other figure.

Just as Escher deciphered the mathematics behind the geometric shapes of the Moorish masters, this exploration encourages students to discover the mathematical secrets behind Escher's tessellations of recognizable animal forms. Although students may at first perceive Escher's works to be art unrelated to mathematics, by the end of the activities they will have the mathematical skills to be Escher sleuths. The first three activities in this set focus students' attention on mathematical aspects of constructing tessellations; the fourth activity reverses the process as students are presented with existing tessellations and asked to identify their mathematical properties.

Activity sheet 1 introduces students to the notion of tessellation in general and asks them to discover which regular polygons tessellate a plane. Although Escher's works extended well beyond the use of single polygonal

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Did You Know?

The Alhambra has many connections to history, literature, and math, expressed as art.

- The Alhambra was built in the thirteenth century. Arab Moors ruled the area until 1492.
- In 1492, Christopher Columbus stood in the Hall of the Ambassadors of the Alhambra as he petitioned Queen Isabella and King Ferdinand to sponsor his trip to the Orient.
- Napoleon's troops used the Alhambra as barracks from 1808 to 1812 and blew up part of the palace as they retreated.
- Washington Irving, American author of *The Legend of Sleepy Hollow,* also wrote *Tales of the Alhambra* in 1832, which stirred pride and passion among the Spanish people to restore the palace to its former beauty.

shapes, we restrict ourselves to tessellations involving one shape—either a square, an equilateral triangle, or a regular hexagon. **Activity sheets 2** and **3** explore two common types of transformations (translation and rotation) used by Escher. Students discover how motifs, or modified polygons, can be created and repeatedly translated or rotated to create a tessellation.

Activity sheet 4 puts students' newfound skills as Escher sleuths into practice as they use their mathematical understanding of tessellations to unravel the underlying properties of selected Escher artwork, identifying the basic shape and type of transformation of each. At the end of these activities, students should have a sense of empowerment as they are able to understand how the great master artist M. C. Escher used mathematics to create his images.

WHICH REGULAR POLYGONS TESSELLATE?

The first step to becoming an Escher sleuth is to understand which regular polygons tessellate and why. Students can be given physical models of regular polygons and, through experimentation, determine which ones tessellate. Often they will describe tessellating shapes as being able to cover the entire plane, or sheet of paper, without gaps or overlaps. In other words, a regular polygon can tessellate if multiple copies can be placed adjacent to one another such that they completely rotate around a vertex point without overlap.

Using **activity sheet 1**, students determine which regular polygons tessellate by examining their interior angles, the angle formed by two adjacent sides of the regular polygon. Partitioning the regular polygon into triangular regions allows students to find the sum of the measures of the interior angles. For instance, as shown in **activity sheet 1**, drawing diagonals from any one vertex of a pentagon divides the pentagon into three nonoverlapping triangular regions. The sum of the interior angles of each triangular region is 180°.

Angle addition leads students to the conclusion that the sum of the interior angles of a pentagon partitioned into three triangular regions is

$$3 \times 180° = 540°.$$

Because regular polygons are equiangular, students can find the measure of each interior angle by dividing the sum of all interior angles by the number of angles. For a regular polygon to tessellate, the measure of each interior angle must be a factor of 360°. Students discover, or confirm earlier experimental findings, that the only regular polygons that tessellate are triangles, squares, and hexagons because their interior angles (60°, 90°, and 120°, respectively) are factors of 360°. Repetitions (6, 4, 3, respectively) of each of these polygons can be arranged around a point without a gap or an overlap.

This activity sets the stage for the entire Escher sleuth experience. Students make the connection that the number of copies of a shape (motif) around each vertex is related to the measure of that shape's interior angles, which in turn reveals its original polygonal form before being modified. This will be a critical insight when students must work backward from an Escher print to discover the original polygon on which the tessellation is based.

TESSELLATIONS FORMED THROUGH TRANSLATION AND ROTATION

Now that students have made a connection between regular polygons that tessellate and the size of the polygons' interior angles, they can begin to analyze Escher's construction of tessellations. Although Escher used a wide range of polygons and transformation techniques, we limit this exploration to his works that are based on one of three regular polygons (square, regular hexagon, or equilateral triangle) and one of two types of transformations (translation or rotation). By modifying sides of a polygon and transforming those sides to others through a translation or rotation, Escher was able to create a pleasing recognizable shape, often resembling the form of an animal. Translations are generally the simplest type of transformation because they do not change the orientation of the resulting figure. In our quest to understand how Escher created his tessellations, we recommend beginning with translations of modified squares and regular hexagons, then moving to rotations of modified equilateral triangles and regular hexagons.

Fig. 2 Modified sides are translated to opposite sides in Escher's *Sketch #105* (*Pegasus*).

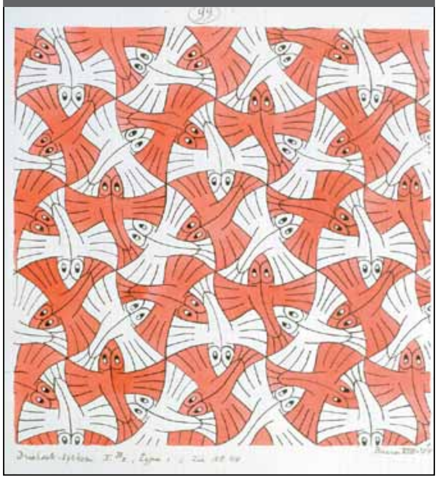

Fig. 3 In *Sketch #99* (*Flying Fish*), Escher may have used rotation to modify an equilateral triangle.

On **activity sheet 2**, students explore tessellations using translations. Given the eventual complexity of the artistic image, the construction of a tessellating tile is more straightforward than might appear. For example, Escher could have used the method illustrated in **activity sheet 2** to form one of his most famous tessellations, *Sketch #105*, which we refer to as *Pegasus* (see **fig. 2**). The steps of the tile creation illustrate how modified sides are translated to opposite sides. Pairs of opposite sides fit like jigsaw pieces and are the focus of question 1 in which students complete the partially labeled array by matching translated sides. This will help them in their role as an Escher sleuth by directing their attention to the relationship between translated sides: Certain pairs of sides always match up.

Question 2 asks students to consider step 6 of the Pegasus creation as they analyze the picture of the Pegasus tessellation and identify how many modified figures, or horses, fit around each vertex. In our experience, students sometimes struggle to tell a "true" vertex (i.e., one that forms part of the original tile) from other vertices formed by the modifications. To avoid confusion, four true vertices are marked in this diagram, each with a large green dot. Students should notice that four horse figures surround each green dot. They should connect this fact to their finding from **activity sheet 1** that tessellations involving squares have four figures around each vertex as $90° \times 4 = 360°$.

Question 3 challenges students to consider a new situation as they are asked to hypothesize how many modified shapes would appear around each vertex if Escher created a tessellation by modifying a regular hexagon instead of a square. Referring to their findings from question 2 and **activity sheet 1**, students should decide that a tessellation involving hexagons must include three hexagons around a vertex point ($3 \times 120° = 360°$). Without a concrete visual image of a hexagon translation, students must rely on their knowledge of angle measures to make this determination. Being able to analyze tessellations based on the number of polygons around a vertex is an important prerequisite for being an Escher sleuth.

Tessellations can also be created through rotation using regular hexagons, squares, or equilateral triangles. The method that is illustrated in **activity sheet 3** demonstrates how Escher may have used rotation to modify an equilateral triangle to create *Sketch #99*, or *Flying Fish* (see **fig. 3**).

Teachers may want to highlight how the various sides of the triangle are transformed. For example, side 1 is modified and rotated clockwise about its endpoint to form side 2. However, the transformation of the third side is different: Half of side 3 is initially modified and then rotated 180° about the midpoint of side 3. Given the partially labeled array of triangles in question 1, students should be encouraged to refer back to the creation of the *Flying Fish* tessellation as they try to label the sides of the triangles that would be rotations of each other.

Question 2 asks students to consider the number of fish that appear around each vertex and how that number reveals the measure of the interior angle of the underlying polygon that Escher used. Students can rely on their experience with **activity sheet 1** to determine that six repetitions of a figure around a vertex indicate that the interior angle of the original regular polygon is 60° ($360° \div 6 = 60°$), which is consistent with the underlying equilateral triangle.

Similar to question 3 found on **activity sheet 2**, question 3 on **activity sheet 3** challenges students to consider how a tessellation based on a rotated regular hexagon could be made. Although the answers are the same (three hexagons surround each vertex), an important distinction is *how* a regular hexagon appears around each vertex point when translated compared with *how* it appears when rotated. Teachers may wish to point out to students that recognizing this distinction will be critical in differentiating tessellations (a) and (c) on **activity sheet 4**.

Together, **activity sheets 2** and **3** direct students' attention to the two key clues for deciphering an Escher tessellation: the number of figures around a vertex point, which reveals the underlying shape; and the relative orientation of the figures, which reveals the type of transformation (a translation or a rotation). Tessellations involving translations are always aligned in the same direction, whereas tessellations based on rotation have figures that point in different directions.

ESCHER SLEUTH IN ACTION

Students are now ready to try their mathematical skills as Escher sleuths. **Activity sheet 4** asks students to take on the role of mathematical detective as they identify the underlying shape (triangle, square, or hexagon) and type of transformation (translation or rotation) of each tessellation. If students experience difficulty with this activity, refer back to previous tessellation examples and ask students to—

1. relate how the number of images that meet at a vertex reveals the underlying polygonal shape; and
2. describe how the orientation of the image is or is not affected by a translation versus a rotation.

In addition to identifying tessellations similar to those encountered in **activities 2** and **3**, students must be able to transfer their Escher sleuth skills to new situations as they identify tessellations created by a rotation of a modified hexagon and by a translation of a modified hexagon. Finally, the Escher sleuths are challenged to find the Escher fake among the six tessellations.

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Schattschneider, Doris. 2004. *M. C. Escher: Visions of Symmetry*. 2nd ed. New York: Abrams.

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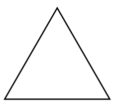
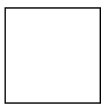
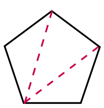
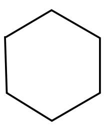
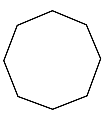
activity sheet 1

Name ______________________________

DISCOVERING WHICH REGULAR POLYGONS TESSELLATE

1. For each regular polygon listed, select one vertex and draw all its diagonals from that point. The pentagon has been completed as an example.

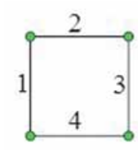
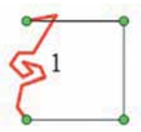
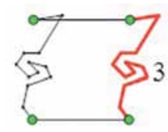
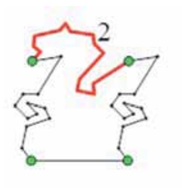
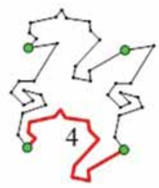
2. Complete the table.

Regular Polygon	Number of Triangles Created	Sum of Interior Angles	Measure of Each Interior Angle	Is 360° Divisible by the Measure of the Interior Angle?	Does This Polygon Tessellate? If So, How Many Repetitions of the Polygon Are Needed around a Rotational Point?
 Triangle	1				
 Square					
 Pentagon	3	$3 \times 180 = 540$			
 Hexagon					
 Octagon					

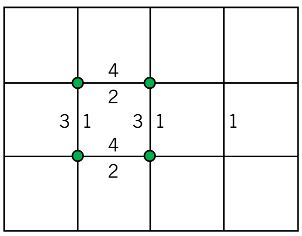
activity sheet 2

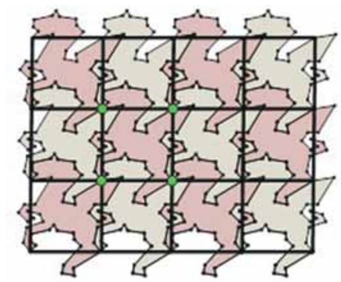
Name ______________________________

TESSELLATIONS USING TRANSLATIONS OF A SQUARE

		
Step 1: Draw (or construct) a square.	Step 2: Create one irregular side (side 1).	Step 3: Translate irregular side 1 to side 3.
		
Step 4: Create second irregular side (side 2).	Step 5: Translate irregular side 2 to side 4.	Step 6: Tessellate.

1. *Extend the pattern.* Examine the steps (above) that M. C. Escher might have used to construct his *Pegasus* tessellation. Using the sides of the squares as labeled above, complete the pattern of labels below to show how a modified square tessellates by translation.





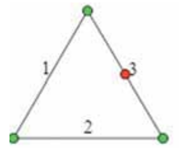
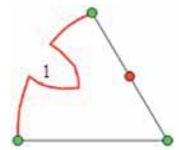
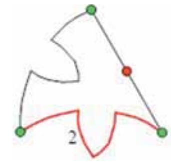
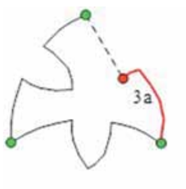
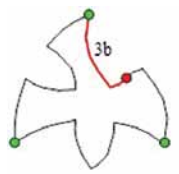
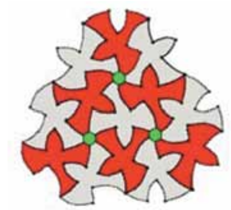
2. *Analyze the tessellation of the modified square.* In the figure above, how many horses appear around each vertex marked with a large dot? How does that number reveal the measure of the interior angles of the initial tile? Which regular polygon has interior angles with this measure?

3. ***Extension:*** Escher created the *Pegasus* tessellation based on a square tile that has been translated. If he created a tessellation based on the translation of a regular hexagon, how many modified images would appear around each vertex point? (Hint: Think about **activity sheet 1**.)

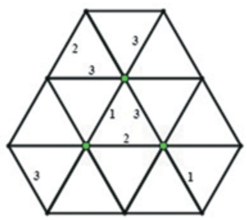
activity sheet 3

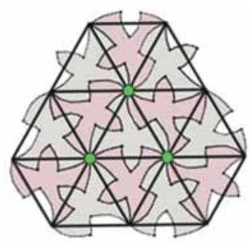
Name ______________________________

TESSELLATIONS OF AN EQUILATERAL TRIANGLE

		
Step 1: Draw (or construct) a triangle with midpoint of side 3.	Step 2: Create one irregular side (side 1).	Step 3: Rotate irregular side to side 2.
		
Step 4: Create irregular edge on half of side 3.	Step 5: Rotate irregular half-edge to rest of side 3.	Step 6: Tessellate!

1. *Extend the pattern.* Examine the steps (above) that M. C. Escher might have used to construct his *Flying Fish* tessellation. Using the sides of the equilateral triangle as labeled above, complete the pattern of labels below to show how a modified triangle tessellates by rotation.





2. *Analyze the tessellation of the modified triangle.* In the figure above, how many flying fish appear around each vertex marked with a large dot? How does that number reveal the measure of the interior angles of the initial tile? Which regular polygon has interior angles with this measure?

3. ***Extension:*** Escher created the *Flying Fish* tessellation based on an equilateral triangular tile that had been rotated. If he created a tessellation based on the rotation of a regular hexagon, how many modified images would appear around each vertex point? (Hint: Use what you learned from **activity sheet 1** and question 2 above.)

activity sheet 4

Name ______________________________

ARE YOU AN ESCHER SLEUTH?

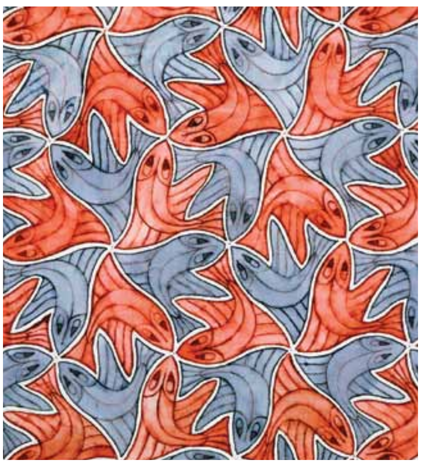
1. Identify the following tessellations by their underlying shape (triangle, rectangle, or hexagon) and type of transformation (translation or rotation).

Type of transformation __________
Underlying shape ______________
(a)

Type of transformation __________
Underlying shape ______________
(b)

Type of transformation __________
Underlying shape ______________
(c)

Type of transformation __________
Underlying shape ______________
(d)



Type of transformation __________
Underlying shape ______________
(e)

Type of transformation __________
Underlying shape ______________
(f)

2. Which of the tessellations a–f was *not* generated by M. C. Escher? How do you know?

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